The constant-Mach-number MHD generator

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It is shown that the optimum design of duct in a magnetohydrodynamic generator is close to the one in which the flow Mach number remains constant. This constant-Mach-number generator is analysed in some detail and it is shown that the optimum Mach number can be defined to within a few percent. For a γ of 1·25, this optimum is near 0·85. For very short ducts, the maximum power output is obtained near matched-load conditions but for rather longer ones maximum total power output is obtained by working as close to short-circuit conditions as is practicable. Against this, the minimum compressor requirements are found by working as close to *open*-circuit conditions as is practicable, and so a compromise must be reached for optimum overall generator design as far as load conditions are concerned. This will probably give an internal ohmic loss in the fluid of about one-third of the total output. Curves are presented which enable the optimum Mach number to be determined with greater precision when the optimum load conditions have been selected.

1. Introduction

During the last two years, a number of papers have been published in which the various thermodynamic parameters in an MHD duct have been obtained under a variety of simplifying conditions. For example, almost all authors ignore heat transfer through the duct walls as well as friction in the duct. In order to solve the equations most authors also make other simplifying assumptions about the nature of the flow. Neuringer (1960) solved the MHD equations numerically for the case of a duct of constant cross-sectional area and for a gas of constant conductivity. Coe & Eisen (1960) extended this treatment to the case of a gas with variable conductivity. Other workers (Way 1960; Sutton 1959; Huth 1961) have obtained solutions for the case where one thermodynamic parameter such as pressure, temperature or velocity remains constant down the duct.

In this paper we obtain a solution of the flow equations for the constant-Mach-number case, and we show that this will give a fair approximation to the design required for minimum duct length, provided that not too much total power is extracted. It is believed that this solution will be close to the case selected for the operation of a practical MHD generator since, for economic reasons, it is desired to extract the MHD power from as short a duct as possible. The heat transfer from an MHD generator will clearly depend on the duct length, and also one of the major costs in the construction of an MHD duct will be the cost of the large magnets necessary to generate large magnetic fields over a considerable volume. It was for these reasons that it was decided to choose the criterion of the maximum electrical power extraction per unit length. It can also be argued that a similar criterion such as maximum power extraction per unit volume should be chosen. This case has been analysed and it can be shown that the results differ very little from those for minimum length.

The MHD generator considered here consists of a stagnation chamber (in which hot gas is produced, e.g. by combustion), an expansion nozzle (to accelerate the gas), and a duct in which MHD interaction occurs to generate electrical power. The gas is at a sufficiently high temperature to be electrically conducting and electrodes are placed in the walls of the duct so that current may flow freely in a direction orthogonal to the axis of the duct and to the magnetic field, as in figure 1.

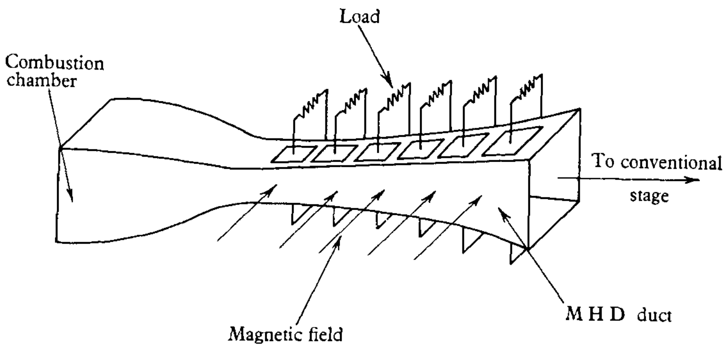

FIGURE 1. MHD duct configuration.

In order to avoid Hall effects, the electrodes are segmented, each pair of opposite electrodes being connected through its own load. The currents in the gas interact with the magnetic field to cause a retarding force and also travel through the electrodes into a load so that electrical power is generated. The enthalpy of the gas is thus converted into electrical power.

Conditions in the stagnation chamber are represented by the subscript 0 and at the entrance to the MHD region by the subscript 1. We assume for simplicity that the generator is sufficiently large for the heat transferred through the walls and the friction between the gas flow and the duct walls to be ignored. It will be shown later how these effects may be included relatively simply in the analysis. End effects and other wall effects are also neglected.

In practical MHD generators the conductivity and hence magnetic Reynolds number will be small. The effect of generated currents on the magnetic field will therefore be ignored and the magnetic field taken to be constant. The flow equations may be written:

force per unit volume
$$\rho v(dv/dx)+(dp/dx)+JB = 0; \qquad (1)$$
power per unit volume
$$\rho v d(c_p T+\tfrac{1}{2}v^2)/dx+JE = 0; \qquad (2)$$
mass flow rate
$$\rho v A = \text{const.}; \qquad (3)$$
equation of state
$$p = \rho c_p T(\gamma-1)/\gamma; \qquad (4)$$
modified Ohm's Law
$$J = \sigma(vB-E);$$
Mach number
$$M^2 = v^2/(\gamma-1)\,c_p T; \qquad (5)$$

here A is the cross-sectional area of the flow, B the magnetic induction, E the transverse electric field, J the current density, T the temperature, c_p the specific heat at constant pressure, ρ the fluid density, σ the electrical conductivity, γ the ratio of the specific heats, p the pressure, v the velocity in the direction x measured along the flow direction, and M the Mach number; the electrical conductivity will be assumed to be a function of pressure and temperature alone, although it is possible to conceive of situations where the electrical currents lead to an elevated electron temperature in the working fluid and the conductivity becomes dependent on the current flowing. Hall effects may be ignored because of the segmented electrodes.

2. Efficiency

The output power per unit length Q is given by

$$\begin{aligned} dQ/dx &= JEA \\ &= -\rho vAd(c_p T + \tfrac{1}{2}v^2)/dx \\ &= -\rho vAd[c_p T\{1+\tfrac{1}{2}(\gamma-1)M^2\}]/dx \\ &= -\rho vAd(c_p TT_{01}/T_1)/dx \\ &= -Q_1 d(T/T_1)/dx, \end{aligned} \tag{6}$$

where $Q_1 = \rho vAC_pT_{01}$ is the total initial power input, and the subscripts 0 and 1 have been defined in §1. (6) integrates to give

$$Q/Q_1 = (T_1-T)/T_1,$$

so that

$$\eta = Q/Q_1 = 1-T/T_1 \tag{7}$$

and

$$T/T_1 = 1-\eta.$$

It is interesting to note from (7) that in the constant-Mach-number case the theoretical efficiency is the Carnot efficiency of a heat engine working between the initial and final *duct* temperatures. The *stagnation* temperature is not involved. Most of the other thermodynamic parameters can now be determined simply in terms of T/T_1 (and hence of the fraction of the power extracted, η) as follows.

3. Other thermodynamic parameters

Eliminating J between (1) and (2) gives

$$\{d(c_pT)/dx\}+\{(1-K)\,d(\tfrac{1}{2}v^2)/dx\}-\{(K/\rho)\,(dp/dx)\} = 0, \tag{8}$$

where

$$K = (JE)/v(JB) = E/vB, \tag{9}$$

and then substituting for ρ from (4) and v^2 from (5) gives

$$(1/T)\,dT/dx = (1/\beta p)\,dp/dx, \tag{10}$$

where

$$\beta = \{1+\tfrac{1}{2}(1-K)\,(\gamma-1)\,M^2\}\,\gamma/(\gamma-1)\,K. \tag{11}$$

For constant β, (10) can be integrated to give

$$T/T_1 = (p/p_1)^{1/\beta} \tag{12}$$

$$= (p/p_1)^{\eta'(\gamma-1)/\gamma}, \tag{13}$$

where

$$\eta' = K/\{1+\tfrac{1}{2}(1-K)\,(\gamma-1)\,M^2\}. \tag{14}$$

(13) is of the form usually associated with non-isentropic expansion of efficiency η'. In this case the non-isentropic efficiency is less than K, the non-isentropic efficiency for the constant velocity case. It should be noted that a sufficient set of conditions for constant β is that K, M and γ should all be constant, and this will be assumed in the subsequent analysis.

From (12),
$$p/p_1 = (T/T_1)^{\beta} = (1-\eta)^{\beta}. \tag{15}$$

From (4),
$$\rho/\rho_1 = (p/p_1)/(T/T_1) = (T/T_1)^{\beta-1} = (1-\eta)^{\beta-1}. \tag{16}$$

From (5),
$$v/v_1 = (M/M_1)(T/T_1)^{\frac{1}{2}} = (T/T_1)^{\frac{1}{2}} = (1-\eta)^{\frac{1}{2}}. \tag{17}$$

From (3),
$$A/A_1 = (\rho_1/\rho)(v_1/v) = (T/T_1)^{-(\beta-\frac{1}{2})} = (1-\eta)^{-(\beta-\frac{1}{2})}. \tag{18}$$

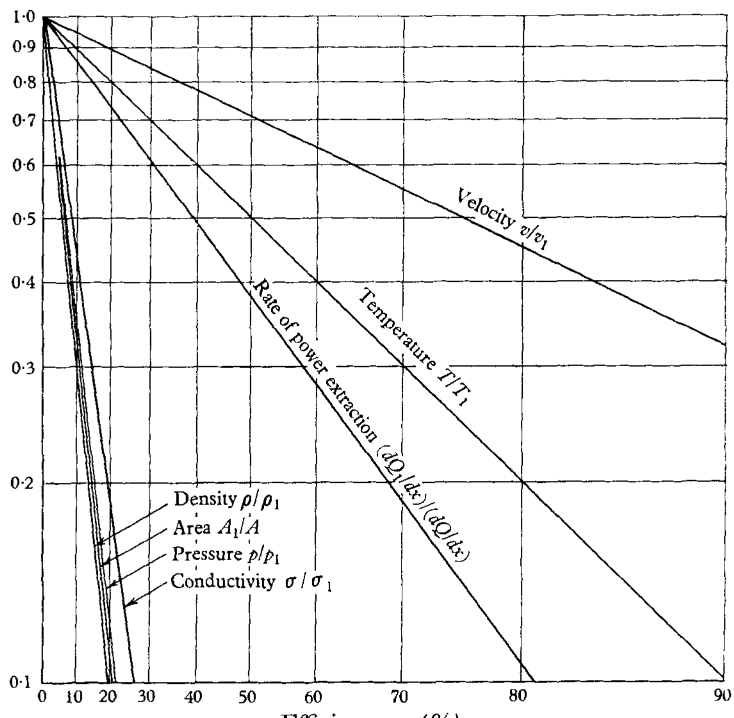

FIGURE 2. Variation of flow parameters with fraction of power extracted.

Equations (7), (15)–(18) are plotted in figure 2 for typical values of γ, K and M. It can be seen that for an efficiency of only 20 % there is a change of about 10 to 1 in pressure and in cross-sectional area. It therefore seems likely that efficiencies will be limited to around that figure by practical considerations.

4. Friction and heat transfer

Friction and other forces (F per unit volume) as well as heat and other forms of energy transfer from the duct (G per unit volume) may be taken into account in a simple way at this stage by suitably redefining K and B. F simply adds to the magnetic retarding force JB in equation (1) while G adds to the electrical power output in equation (2) and the elimination leading to (8) is therefore still valid provided a new value of K is used; thus

$$K' = (JE+F)/v(JB+G). \tag{19}$$

Similarly it will be found in the subsequent analysis that the substitutions leading from (2) to (21) are valid provided

$$JE+G = \sigma v^2 B'^2 K'(1-K'). \tag{20}$$

The analysis may then proceed in terms of the new parameters K' and B' where they, rather than K and B, are held constant. It can therefore be seen that friction, heat transfer, etc., may all be included in this analysis provided that the load and the magnetic field are suitably varied along the duct.

5. Duct length

The length of duct required to extract a fraction η of the total enthalpy can now be found in closed form. By (2)

$$d(c_p T + \tfrac{1}{2}v^2)/dx = -JE/\rho v, \tag{21}$$

so that

$$d(c_p T T_{01}/T_1)/dx = -\sigma v B^2 K(1-K)/\rho,$$

$$d(T/T_1)/dx = -K(1-K)(\sigma/\sigma_1)(v/v_1)(\rho_1/\rho)/x_0, \tag{22}$$

where

$$x_0 = c_p T_{01}\rho_1 v_1/\sigma_1 v_1^2 B^2$$
$$= (Q_1/A_1)/\sigma_1 v_1^2 B^2.$$

Then (16) and (17) with (21) give

$$\int_1^{T/T_1} \frac{\sigma_1}{\sigma}\left(\frac{T}{T_1}\right)^{\beta-\frac{3}{2}} d\frac{T}{T_1} = -K(1-K)\int_0^{x/x_0} d\frac{x}{x_0}$$
$$= -K(1-K)(x/x_0). \tag{23}$$

If σ is a known function of p and T, and hence by (12) of T alone, (23) is integrable.

6. Constant conductivity

If the gas conductivity is held constant regardless of the state of the gas (e.g. by non-thermal ionization) equation (23) can be integrated directly to give an analytic expression for the length

$$x/x_1 = 1-(T/T_1)^{\beta-\frac{1}{2}} = 1-(1-\eta)^{\beta-\frac{1}{2}},$$

where

$$x_1 = x_0/K(1-K)(\beta-\tfrac{1}{2}). \tag{24}$$

In principle it is possible to extract all the energy in the length x_1 to achieve 100 % efficiency. Unfortunately β is normally large, say between 5 and 10, so that the last half of the power should be extracted in the last hundredth of the duct length, while even to extract half the power, the duct cross-sectional area must increase 100-fold along its length. Thus the efficiency is likely to be limited by practical considerations.

7. Variable conductivity

If the ionization is produced thermally, the conductivity is a complicated function of temperature and pressure, even assuming that equilibrium is reached almost instantaneously at each stage. Provided this function is known, however (e.g. from experimental data) equation (23) can be integrated numerically to give the length of the generator. For the present purpose, an analytical expression is to be preferred and may be obtained by using a simple approximation for σ, such as a power law

$$\sigma = (sT)^y p^{-z},$$

with s a dimensional constant. Locally,

$$y = \{d(\ln\sigma)/d(\ln T)\}_p, \quad -z = \{d(\ln\sigma)/d(\ln p)\}_T,$$

and over ranges of interest the best values are shown in table 1.

Conductivity (mho/m)	y	z	s (T in °K, p in atmospheres)
$10 > \sigma > 0{\cdot}1$	13	0·5	1/2,000
$100 > \sigma > 1$	10	0·4	1/2,000

TABLE 1

Thus
$$\sigma/\sigma_1 = (T/T_1)^y (p/p_1)^{-z} = (T/T_1)^{y-\beta z}$$
$$= (1-\eta)^{y-\beta z}. \tag{25}$$

(23) then becomes
$$\int_1^{T/T_1} \left(\frac{T}{T_1}\right)^{\omega-1} d\frac{T}{T_1} = -K(1-K)\frac{x}{x_0}, \tag{26}$$

where
$$\omega = (\beta-\tfrac{1}{2}) - (y-\beta z), \tag{27}$$

so that
$$x/x_2 = 1-(T/T_1)^\omega \quad (\omega \neq 0)$$
$$= 1-(1-\eta)^\omega, \tag{28}$$

where
$$x_2 = x_0/K(1-K)\,\omega. \tag{29}$$

Equation (28) is plotted in figure 3. (28) may be rewritten as

$$\eta = 1-\{1-xK(1-K)\,\omega/x_0\}^{1/\omega}. \tag{30}$$

Although some of the separate terms in the expression for ω are large, they differ in sign and the total value of ω may be small positive or negative or even zero. For positive values of ω all the power can in principle be extracted in a finite length x_2. For negative values of ω, x_2 is negative and

$$\eta = 1-\{1+xK(1-K)\,|\omega|/x_0\}^{-1/|\omega|},$$

so that all the power can in principle be extracted in an infinite length. For the singular case in which ω vanishes, (26) integrates to give

$$\eta = 1-\exp\{-xK(1-K)/x_0\},$$

a result which may also be obtained by taking the limit of (30) as ω tends to zero. Again, all the power can in principle be extracted in an infinite length.

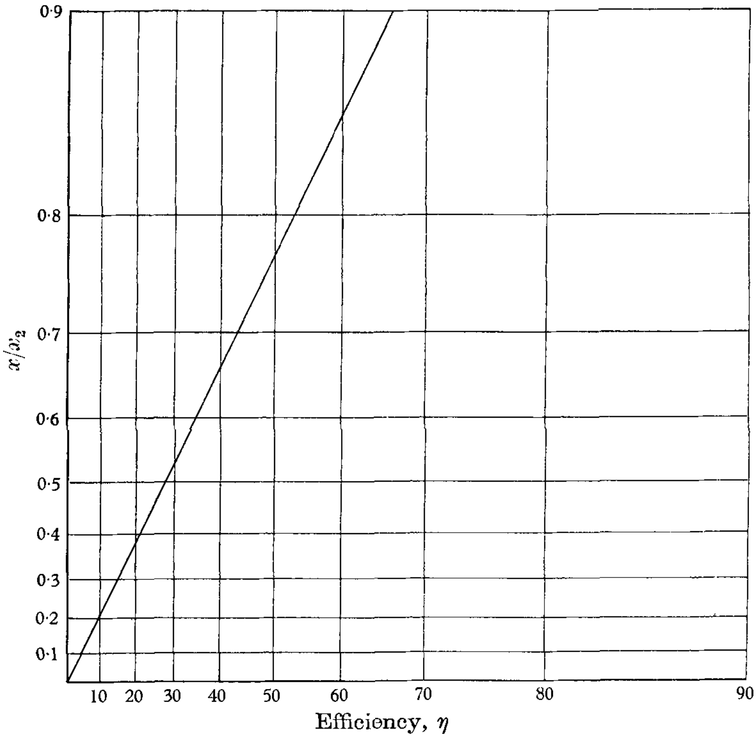

FIGURE 3. Distance *vs* fraction of power extracted.

8. Local optimum conditions

The power output per unit length at any point along the duct is

$$JEA = \sigma v^2 B^2 AK(1-K), \tag{31}$$

which is optimized locally by choosing the load so that K is equal to $\frac{1}{2}$, i.e. by matching the load to the internal resistance of the plasma. This is true at all points along the duct, and the same condition holds for the optimum power per unit volume, JE. In a similar way a locally optimum Mach number can be chosen. In terms of the local stagnation conditions

$$\sigma = \sigma_0(T/T_1)^y (p/p_0)^{-z}, \tag{32}$$

$$p/p_0 = (T/T_0)^{\gamma/(\gamma-1)}, \tag{33}$$

$$\rho/\rho_0 = (T/T_0)^{\gamma/(\gamma-1)-1}, \tag{34}$$

and

$$\tfrac{1}{2}v^2 = c_p(T_0 - T), \tag{35}$$

so that the power per unit length is

$$JEA = \{\sigma_0(2c_p T_0)^{\frac{1}{2}} \rho v AB^2/\rho_0\} (T/T_0)^{\gamma} (1 - T/T_0)^{\frac{1}{2}} K(1-K), \tag{36}$$

where

$$\lambda = (y+1) - (z+1)\gamma/(\gamma-1), \tag{37}$$

or, since

$$T_0/T = \{1 + \tfrac{1}{2}(\gamma-1)M^2\} = (1+X) \quad \text{say}, \tag{38}$$

$$JEA \propto X^{\frac{1}{2}}/(1+X)^{\lambda+\frac{1}{2}} \tag{39}$$

for any given stagnation conditions along the duct and for *any* fixed value of K. Straightforward logarithmic differentiation shows that the optimum value of X for which the maximum rate of power extraction occurs is

$$X_{\text{opt}} = \tfrac{1}{2}/\lambda, \tag{40}$$

which gives an optimum Mach number of

$$M_{\text{opt}} = \{(y+1)(\gamma-1)-(z+1)\gamma\}^{-\frac{1}{2}}. \tag{41}$$

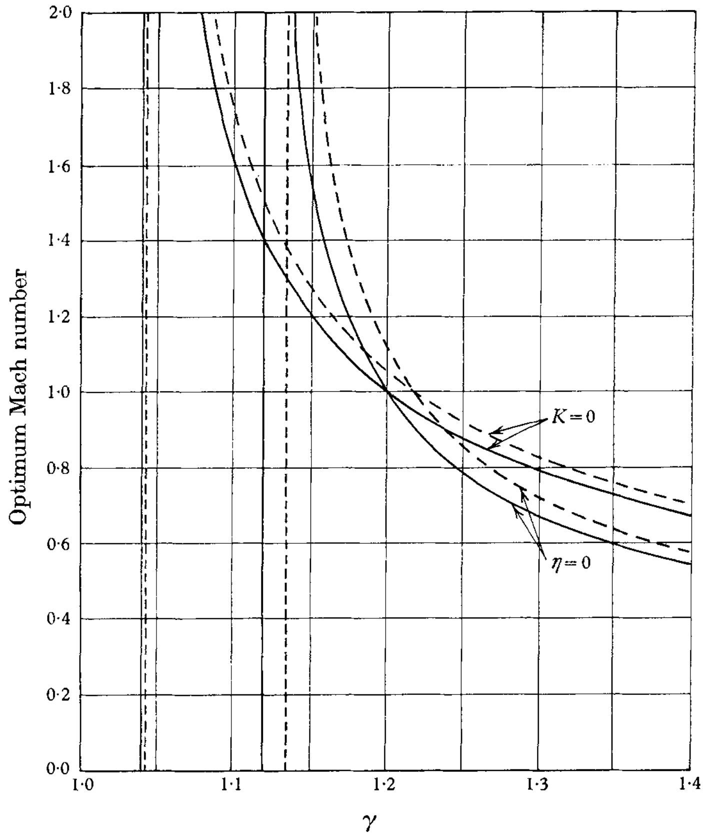

FIGURE 4. Variation of optimum Mach number with γ. ——, Minimum length; ---, minimum volume.

For typical values of the parameters ($y = 13$, $z = \frac{1}{2}$, $y = 1{\cdot}25$) the optimum Mach number is 0·78, but it is in fact a function of γ as shown in figure 4. Thus at any point along the duct this is the value of Mach number required to give a maximum rate of power extraction, independent of the stagnation conditions at that point at least to within our approximation for the conductivity and independent of the load conditions or K.

The fact that the optimum Mach number is the same (for a given γ) at all points along the duct independent of the locally varying stagnation conditions is interesting, since it would appear at first sight that a constant Mach number should give the minimum duct length for an MHD generator. However, four points must be borne in mind. First, only an *approximate* expression for the conductivity has been used. If a more accurate expression involving exponentials is used, it has been found (Ralph 1962) that the locally optimum Mach number *varies* with stagnation temperature as power is extracted along the duct. Secondly, to extract a reasonable amount of power, a wide range of variation in the physical properties is required. Even to extract 20 % of the power requires a change in the duct area and in the pressure of about 10 to 1 and the quasi-one-dimensional approximation may start to break down as well as that for conductivity. Third, γ may well vary appreciably with temperature and pressure along the duct, and the analysis then fails. Lastly, the quickest way down a hill is not necessarily the line of steepest descent, and to extract a given fraction of the total power in the shortest possible *overall* length, the total length x must be minimized rather than the local incremental length at each stage. Thus it is not strictly true that constant Mach number necessarily gives the shortest duct design. However, it is shown in Appendix 1, particularly for low efficiencies, that the constant-Mach-number generator will not be very different from the shortest design.

9. Optimum load conditions

The optimum design of an MHD generator will take account of the cost or weight of the various items of equipment such as the magnet, the compressors and the MHD duct itself, as well as the overall efficiency of the system. As far as the constructional costs of the magnet and the MHD duct are concerned it seems clear that the optimum design will be close to that for minimum length or volume, and this is also true for the loss of efficiency due to heat transfer through the walls which may be considerable if cooled walls are used. It is shown in Appendices 1 and 2 that the value of K which gives an overall minimum length or volume of duct is always less than $\frac{1}{2}$ and is in fact mathematically zero for reasonable efficiencies, although this is obviously not physically realizable and can only be regarded as a limiting case.

It has already been remarked that the relationship (13) between p and T is of the form usually associated with non-isentropic expansion of efficiency η', and in normal aerodynamic flows the increase in entropy is caused by friction, which converts high-grade (kinetic) energy back into thermal energy. In our case we have assumed that the retarding force is mainly electromagnetic and is JB per unit volume, and high-grade (electrical) energy is converted back into thermal energy by straightforward ohmic heating of the fluid. As in the aerodynamic case, this increase in entropy is undesirable since an increased pressure drop must be provided which will require larger compressors and will need greater compressor driving powers which must be subtracted from the output of the generator. For minimum compressor requirements η' should be as high as possible, and this occurs when $K = 1$. Again this represents a limit that is not physically realizable (although it should be remarked that conventional alternators or batteries approach this limit very closely, values of K of 0·997 being typical).

It can therefore be seen that the optimum value of K will represent a compromise between the requirements of minimum duct size ($K \to 0$) and minimum compressor capacity ($K \to 1$). This compromise cannot be determined with great accuracy until the detailed economics of duct and magnetic construction have been worked out, and since no duct has yet been built with an interesting lifetime and no large superconducting magnet (as is usually proposed for large-scale generators) has yet been demonstrated, this is not at present possible. However, such estimates as have been made appear to indicate that a K of at least $\frac{3}{4}$ will be necessary.

10. Optimum Mach number

It is shown in Appendix 1 that the average power per unit length of duct is a function of both Mach number and load conditions, equation (44). For any given value of K, this function can be differentiated logarithmically to obtain the

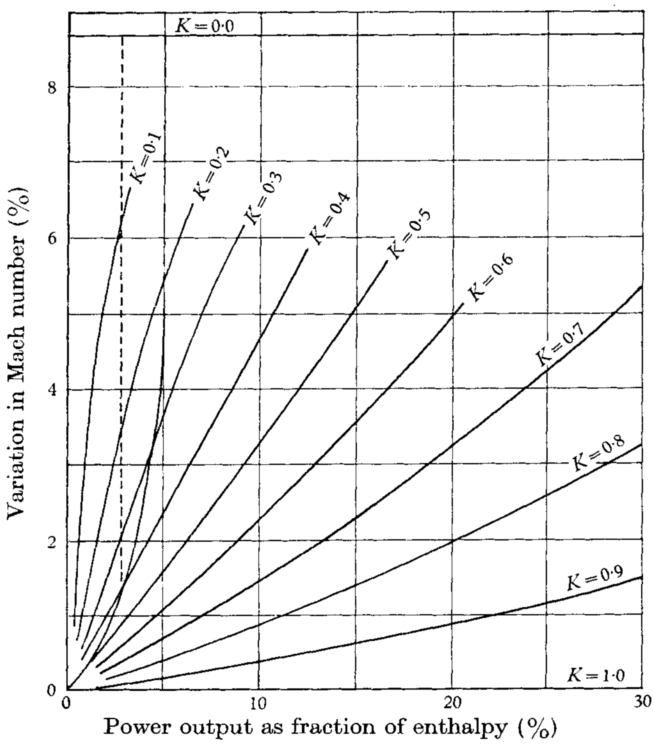

FIGURE 5. Variation of optimum Mach number (for minimum length) with fraction of power extracted, for various load conditions. $\gamma = 1{\cdot}25$.

condition for optimum Mach number giving maximum average power per unit length as in (47) (see Appendix 1). For $\eta = 0$ the optimum is of course that given by (40) regardless of the value of K. For generators of non-zero length the optimum value of X lies above $1/2\lambda$. The variation in optimum Mach number expressed as an increase over the value given by (41) is shown in figure 5 and it can be seen that the total variation is only a few percent for the range of interest. The optimum Mach number is 0·78 when $K = 1$ and 0·85 when $K = 0$. Thus, regardless of the optimum value of K, the optimum Mach number for minimum duct length can be roughly established to within $\pm 5\,\%$ say, provided that the ratio γ of the specific heats is known. For a more accurate determination of the optimum Mach number, the appropriate value of K must be specified by considering compressor size and power, duct cost, etc., and then figure 5 may be used.

In a similar way it is shown in Appendix 2 that the value of X giving maximum power per unit volume is $1/\mu$ for $K = 1$ and $1/(\mu - 1)$ for $K = 0$, where

$$\mu = y - z\gamma(\gamma - 1), \tag{37a}$$

which for typical values ($\gamma = 1{\cdot}25$) gives Mach numbers in the range 0·87 for $K = 1$ to 0·92 for $K = 0$, a variation of only $\pm 2\frac{1}{2}\,\%$. The Mach number can therefore be even more closely specified in this case, while the total variation in Mach number between this and the minimum-length case is still only $\pm 8\,\%$.

11. Conclusions

When a conducting gas moves through a magnetic field the rate at which electrical energy can be extracted (per unit length or per unit volume) is expressible in terms of the Mach number of the flow and the extraction efficiency K (the ratio of the electrical power output to the total power generated). The output can therefore be optimized for minimum length or minimum volume.

At any point along the flow maximum electrical output is obtained with an optimum Mach number which is the same at all points along the flow, if a power-law approximation is used for the variation of conductivity with pressure and temperature and γ is assumed constant. It might therefore seem that the optimum design for minimum size of an electrical generator working on these principles should be for a constant Mach number, which would be just below sonic in a practical case.

The constant-Mach-number generator has therefore been investigated and it is found that while the maximum specific power output occurs locally under matched load conditions ($K = \frac{1}{2}$), the maximum *average* specific power output occurs for conditions closer to short circuit ($K < \frac{1}{2}$). It is of academic interest to note that for moderate outputs the maximum *average* specific power occurs theoretically *at* short circuit ($K = 0$). Against this, the minimum compressor requirements are found under open circuit conditions ($K = 1$) and so a compromise must be reached for optimum generator design which will probably give $K \simeq \frac{3}{4}$. The optimum Mach number is limited in range and can therefore be specified to within a few percent regardless of the above considerations concerning the value of K. For a γ of 1·25, $M_{\text{opt}} = 0{\cdot}85$ to within $8\,\%$. Curves are presented which will enable M_{opt} to be determined with greater precision when the optimum value of K has been selected.

Appendix 1. Power per unit length

From (28) and (29),

$$x = x_0[1 - (1-\eta)^{\omega}]/K(1-K)\,\omega. \tag{42}$$

By substituting (31) and (36) into (22) it can be seen that

$$x_0 = [\sigma_0(2/c_p T_{01})^{\frac{1}{2}} B^2/\rho_0]^{-1}\,(T_{01}/T_1)^{\lambda}\,[1 - T_1/T_{01}]^{-\frac{1}{2}}. \tag{43}$$

Now the overall average power per unit length for any given output ηQ_1 is $\eta Q_1/x$, and by (42) and (43)

$$\eta Q_1/x = Pf(X, K), \tag{44}$$

where

$$P = Q_1[\sigma_0(2/c_p T_{01})^{\frac{1}{2}} B^2/\rho_0] \tag{45}$$

is independent of K and X and

$$f(X,K) = [X^{\frac{1}{2}}/(1+X)^{\lambda+\frac{1}{2}}]\,K(1-K)\,\omega\eta[1-(1-\eta)^{\omega}]. \tag{46}$$

To find the optimum Mach number for any specified value of K, (46) may be differentiated logarithmically, thus

$$0 = [(1/2X)-(\lambda+\tfrac{1}{2})/(1+X)]+[1+(1-\eta)^{\omega}\ln(1-\eta)^{\omega}/\{1-(1-\eta)^{\omega}\}] \times (1/\omega)\,[(z+1)\,\gamma/(\gamma-1)]\,[(1/K)-1]. \tag{47}$$

To first order in η, this gives

$$X \simeq (1/2\lambda)\,[1+\eta\omega'(1+1/2\lambda)\,(1/2\lambda)],$$

where

$$\omega' = d\omega/dX = (1+z)\,[\gamma/(\gamma-1)]\,[(1-K)/K],$$

but for small K or large η the optimum values of X must be obtained numerically. The results are shown in figure 5.

It is of interest to find the load conditions for minimum length, although these may not correspond to an overall optimum when account is taken of compressor requirements. Now if $f(X, K)$ has a local maximum with respect to both X and K it may be found by differentiating logarithmically with respect to these variables, thus

$$0 = [(1/K)-1/(1-K)]-[1+(1-\eta)^{\omega}\ln(1-\eta)^{\omega}/\{1-(1-\eta)^{\omega}\}] + (1/\omega)\,[(z+1)\,\gamma/(\gamma-1)]\,[(1+X)/K^2]. \tag{48}$$

Eliminating η between (47) and (48), these reductions reduce after a little manipulation to the simple relation $X = [4K+2(\lambda-1)]^{-1}$ between the optimum values of X and K. Using this relation to determine η it is found that a local maximum exists for values of η up to 5 % as shown in figure 5 for typical values of the parameters. For sufficiently low values of η

$$f(X,K) \simeq [X/^{\frac{1}{2}}(1+X)^{\lambda+\frac{1}{2}}]\,K(1-K). \tag{49}$$

However, it may be possible to find a region away from the local maximum where f is in fact larger than the local maximum value. Thus for any non-zero η the limiting value as K vanishes is

$$f(X,0) = [X^{\frac{1}{2}}/(1+X)^{\lambda-\frac{1}{2}}]\,\eta(1+z)\,\gamma/(\gamma-1), \tag{50}$$

and this value will be greater than the value of $f(X, K)$ given by (49) provided that

$$\eta > K(1-K)\,(\gamma-1)/\gamma(1+z)\,(1+X). \tag{51}$$

Choosing typical values, it is found that for overall efficiencies greater than about 3 % it is theoretically better to work with a zero value of K, i.e. at short-circuit conditions. The optimum value of X in that case is

$$X_{\text{opt}} = \tfrac{1}{2}/(\lambda-1) \tag{52}$$

and

$$M_{\text{opt}} = [y(\gamma-1)-(z+1)\,\gamma]^{-\frac{1}{2}}. \tag{53}$$

For typical values of the parameters this gives the optimum Mach number of 0·85 as shown in figure 5.

This result is rather surprising since a zero value of K corresponds to a zero value of output per unit volume for maximum average power per unit length. The paradox is resolved by observing that when $K = 0$ equation (28) shows that for any length less than x_2 no power is extracted ($\eta = 0$), while for any fraction $\eta > 0$, $x = x_2$, so that all the power is extracted in an infinitesimal length at the end of the duct x_2. The power output per unit volume is zero at all positions up to x_2 where it becomes infinite like $K \exp(-\text{const.}/K)$. However, the total power there is finite (equal to Q_1) so that although the cross-sectional area becomes infinite like $\exp(\text{const.}/K)$ the volume there does not, but is of order $K^{-1}\exp(\text{const.}/K)$ to keep the total power finite. The power per unit length is zero at all points except x_2 where it is infinite of order $K^{-1}\exp(\text{const.}/K)$ in such a way that the power extracted over an infinitesimal length at x_2 is finite (equal again to Q_1). For such rapid duct expansions the quasi-one-dimensional approximation will no longer be valid and the assumed variation of conductivity will not apply over such large ranges of pressures, so that this short-circuit condition cannot be achieved in practice. However, it will obviously be best for minimum length to work with the smallest value of K that can be achieved. In this respect the situation is the converse of the normal generator or battery case, in which the internal resistance is made as small as possible compared with the load (i.e. $K \simeq 1$), and it is best to work as near open circuit as possible. It may be noted that $K = 0$ is also found to be optimum condition for higher efficiencies in the constant temperature generator (Swift-Hook 1961).

Appendix 2. Power per unit volume

The above optimization procedure for power output was carried out with respect to length. A similar procedure may be carried out with respect to any other parameter such as volume. The analysis is analogous to that already carried out in equations (31) to (67). Thus, the power per unit volume at any point along the duct is

$$JE = \sigma v^2 B^2 K(1-K), \tag{31a}$$

and as already pointed out this is optimized locally by choosing the load so that K is equal to one-half, as before. Using (32) to (35) with (38)

$$JE \sim X/(1+X)^{\mu+1}, \tag{39a}$$

where

$$\mu = y - z\gamma/(\gamma-1). \tag{37a}$$

The optimum value of X is

$$X_{\text{opt}} = 1/\mu, \tag{40a}$$

giving

$$M_{\text{opt}} = [2/\{y(\gamma-1)-z\gamma\}]^{-\frac{1}{2}}. \tag{41a}$$

For typical values of the parameters the optimum Mach number is 0·87 and its variation with γ is shown in figure 4. From (21) we have that

$$\int_1^{T/T_1} \frac{\sigma_1}{\sigma}\left(\frac{v_1}{v}\right)^2 d\frac{T}{T_1} = -\frac{K(1-K)}{x_0 A_1}\int_0^x A\,dx = -K(1-K)\frac{V}{x_0 A_1}, \tag{23a}$$

where the total volume V is $\int A\,dx$. By (17) and (25),

$$V = \frac{x_0 A_1}{K(1-K)} \int_1^{T/T_1} \left(\frac{T}{T_1}\right)^{\xi-1} d\frac{T}{T_1} \tag{26a}$$

$$= x_0 A_1 [1-(1-\eta)^{\xi}]/\xi K(1-K) \tag{42a}$$

by (7), where

$$\xi = \beta z - y. \tag{27a}$$

Hence the overall average power per unit volume is

$$\eta Q_1/V = P_V g(X,K), \tag{44a}$$

where

$$P_V = 2c_p T_{01} \sigma_0 B^2 \tag{45a}$$

independent of X and K and

$$g(X,K) = [X/(1+X)^{\mu+1}]\,K(1-K)\,\xi\eta/[1-(1-\eta)^{\xi}]. \tag{46a}$$

For sufficiently small values of η,

$$g(X,K) \simeq \{X/(1+X)^{\mu+1}\}\,K(1-K), \tag{49a}$$

but as before it may be possible to find a region away from the local maximum where g is in fact larger than the local maximum value given where $K = \frac{1}{2}$, $X = 1/\mu$. Thus for any non-zero η the limiting value of $g(X, K)$ as K vanishes is

$$g(X,0) = [X/(1+X)^{\mu}]\,\eta z\gamma/(\gamma-1), \tag{50a}$$

and this value will be greater than the value of $g(X,K)$ given by (49a) provided that

$$\eta > K(1-K)(\gamma-1)/\gamma z(1+K). \tag{51a}$$

Choosing typical values as before it is found that for overall efficiencies greater than about 9% it is theoretically better to work with a zero value of K, i.e. at short circuit conditions, for maximum power per unit volume. The optimum value of X in that case is

$$X_{\text{opt}} = 1/(\mu-1) \tag{52a}$$

and

$$M_{\text{opt}} = [2/\{(y-1)(\gamma-1)-z\gamma\}]^{-\frac{1}{2}}. \tag{53a}$$

For typical values of the parameters this gives the optimum Mach number of 0·92 and the variation with γ is shown in figure 4.

The work was carried out at the Central Electricity Research Laboratories and is published by permission of the Central Electricity Generating Board.

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